Probability HW1 Solution

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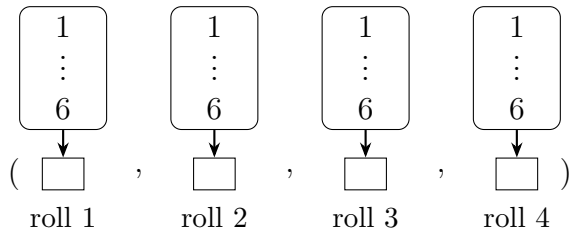
Problem 2.

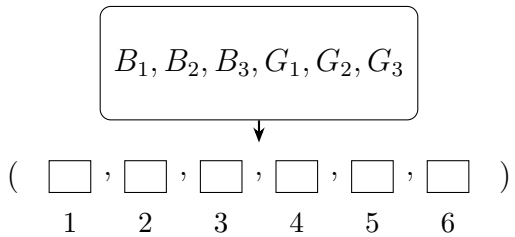


Each roll has six possible outcomes $(1, 2, 3, 4, 5, \text{or } 6)$, and the four rolls are not related to each other. Therefore, the outcome is a list of length four. By the multiplication principle, there are $6^4 = 1296$ results.

Problem 7.

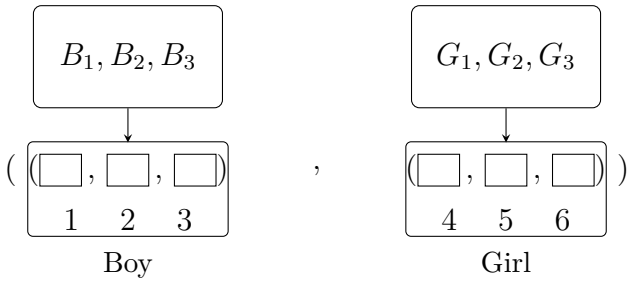
Let the boys be B_1, B_2, B_3 and the girls be G_1, G_2, G_3.

7(a)



Since all six people are distinct, so we may simply permute six objects. Therefore, there are $6! = 720$ ways.

7(b)



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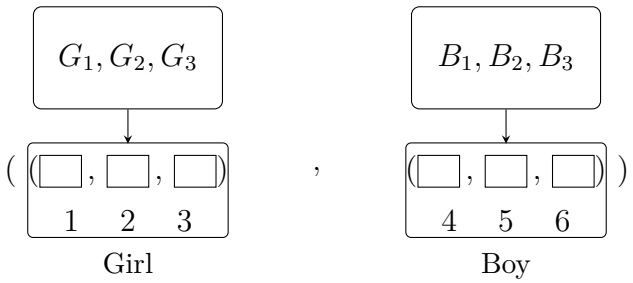
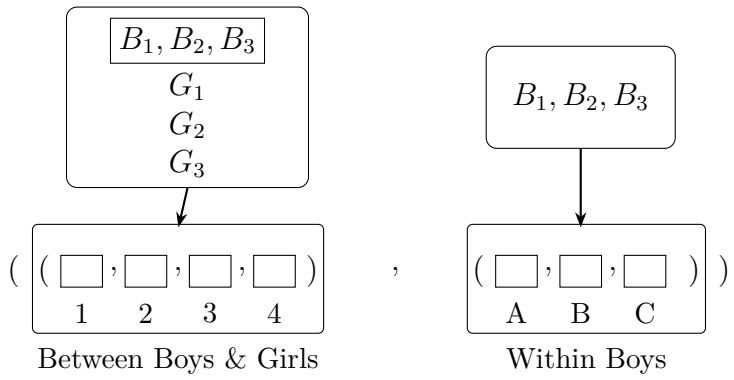


Treat the three boys as one block and the three girls as another block. The two blocks can be ordered in 2! ways, while the people inside each block can be ordered in 3! ways. Therefore, there are $2! \times 3! \times 3! = 72$ ways.

7(c)

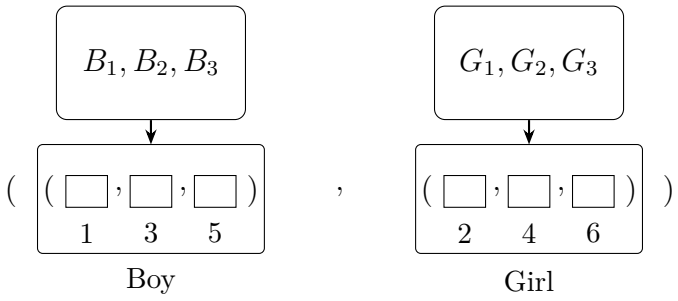


Treat the three boys as one new object. We then arrange four objects: the boy block and the three individual girls. After that, arrange the boys inside their block. Therefore, there are $4! \times 3! = 144$ ways.

7(d)

The sexes must alternate. There are only two possible gender patterns:

$BGBGBG$ or $GBGBGB$.



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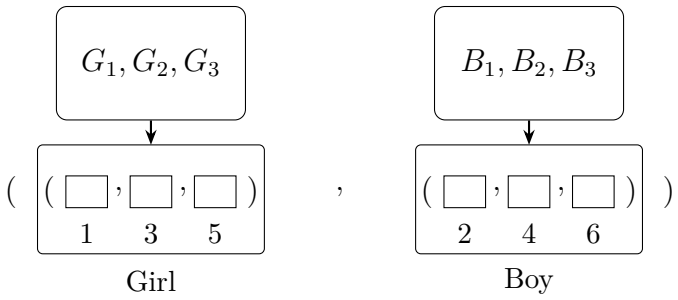
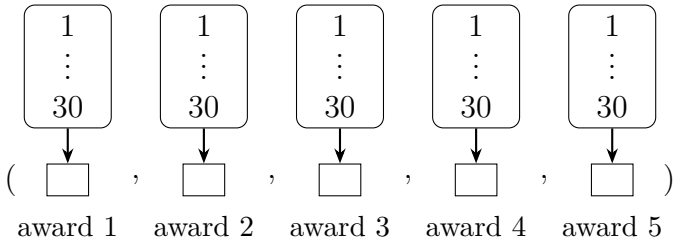


For either pattern, the boys can be arranged in 3! ways and the girls in 3! ways. Therefore, there are $2 \times 3! \times 3! = 72$ ways.

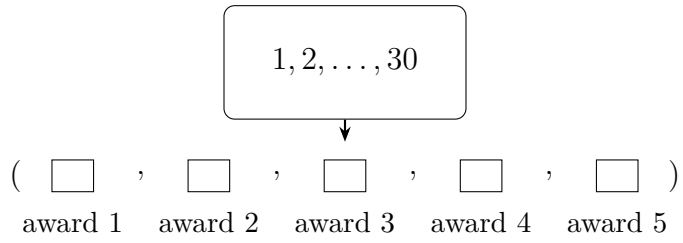
Problem 14.

14(a)



For each award there are 30 possible recipients, and repetition is allowed. Therefore, there are $30^5 = 24,300,000$ possible outcomes.

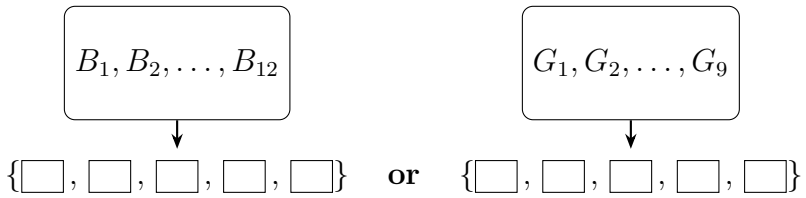
14(b)



Now the five recipients must be distinct. It forms a permutation of length 5. Therefore, there are $30 \times 29 \times 28 \times 27 \times 26 = 17,100,720$ possible outcomes.

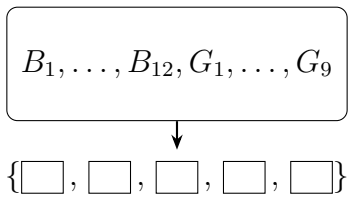
Problem 18.

18(a)



There are two disjoint cases: choose all five players from the 12 boys, or choose all five from the 9 girls. By the addition principle, there are $\binom{12}{5} + \binom{9}{5} = 792 + 126 = 918$ possible choices.

18(b)

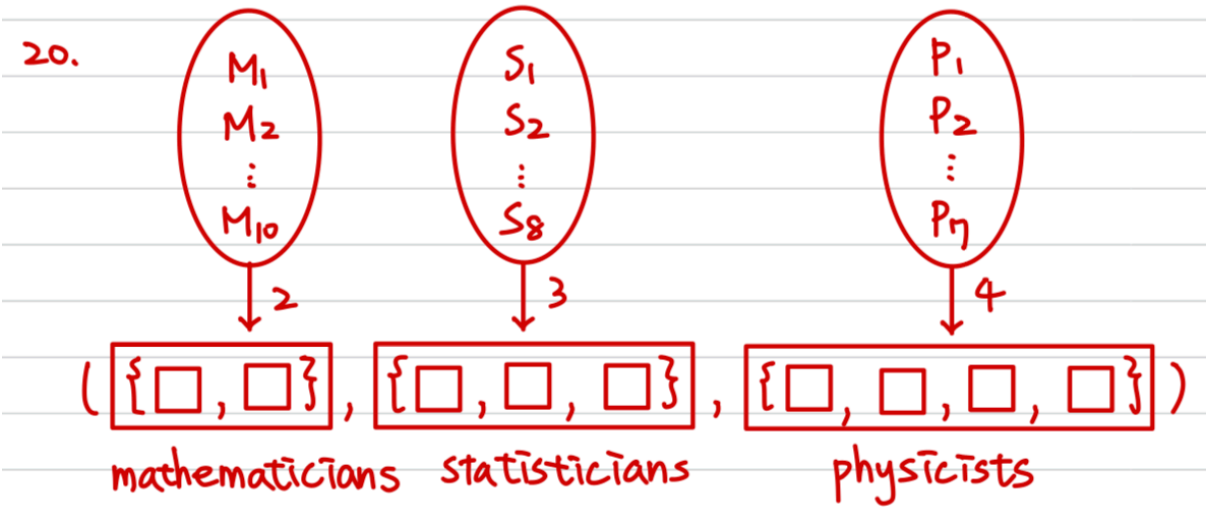


We may use the complement to count. First, choose any five players from all 21 students. Then, subtract the teams whose players are all of the same gender (see 18(a)). Therefore, there are $\binom{21}{5} - \binom{12}{5} - \binom{9}{5} = 20,349 - 918 = 19,431$ possible choices.

Problem 20.

Let M, S, P be mathematicians, statisticians, and physicists, respectively. Each group is independent, so we can apply Multiplication Principle. The answer is

$$\binom{10}{2}\binom{8}{3}\binom{7}{4} = 88200 \text{ possible teams}$$

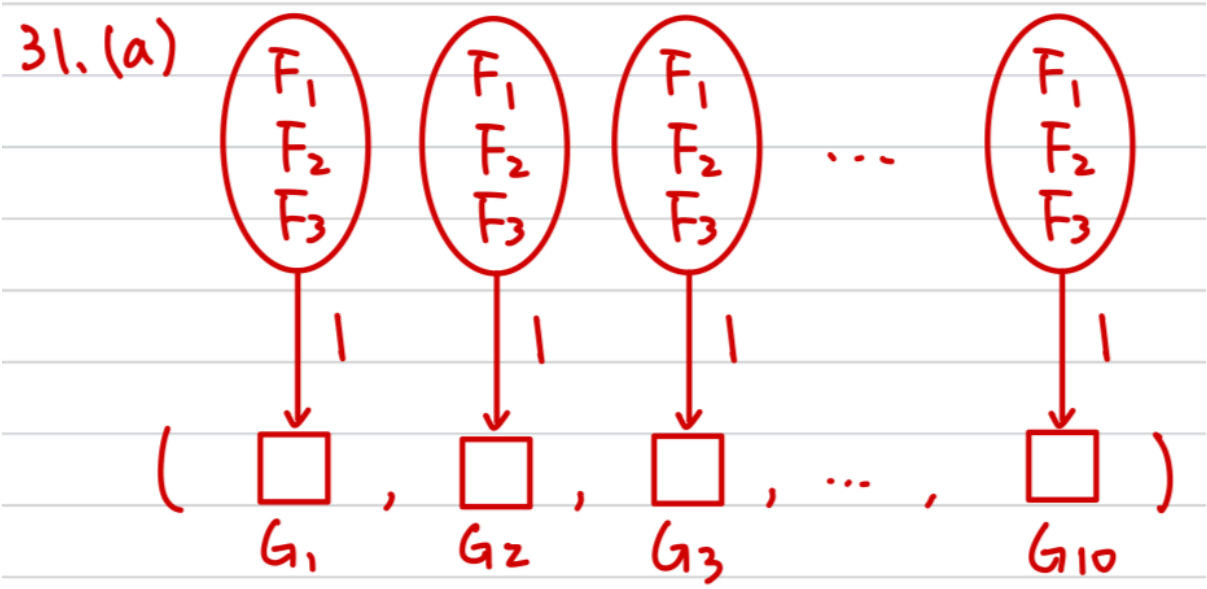


Problem 31.

31.(a)

Let F_1, F_2, F_3 be the 3 friends, $G_1, ..., G_{10}$ be the 10 gifts. Each gift can be given to any one of the 3 friends. Therefore, there are 3 options for each gift. By Multiplication Principle, the answer is

$$3^{10} = 59049$$



31.(b)

Since everyone must receive at least 3 gifts, we can divide the 10 gifts into 3 piles, with 4, 3, and 3 gifts in each pile.

We first draw 4 gifts and decide who gets them. The number of possible combinations is

$$\binom{10}{4} \times 3$$

Then, choose 3 gifts from the remaining 6 gifts and give them to one of the 2 friends. The remaining 3 gifts are then given to the other friend.

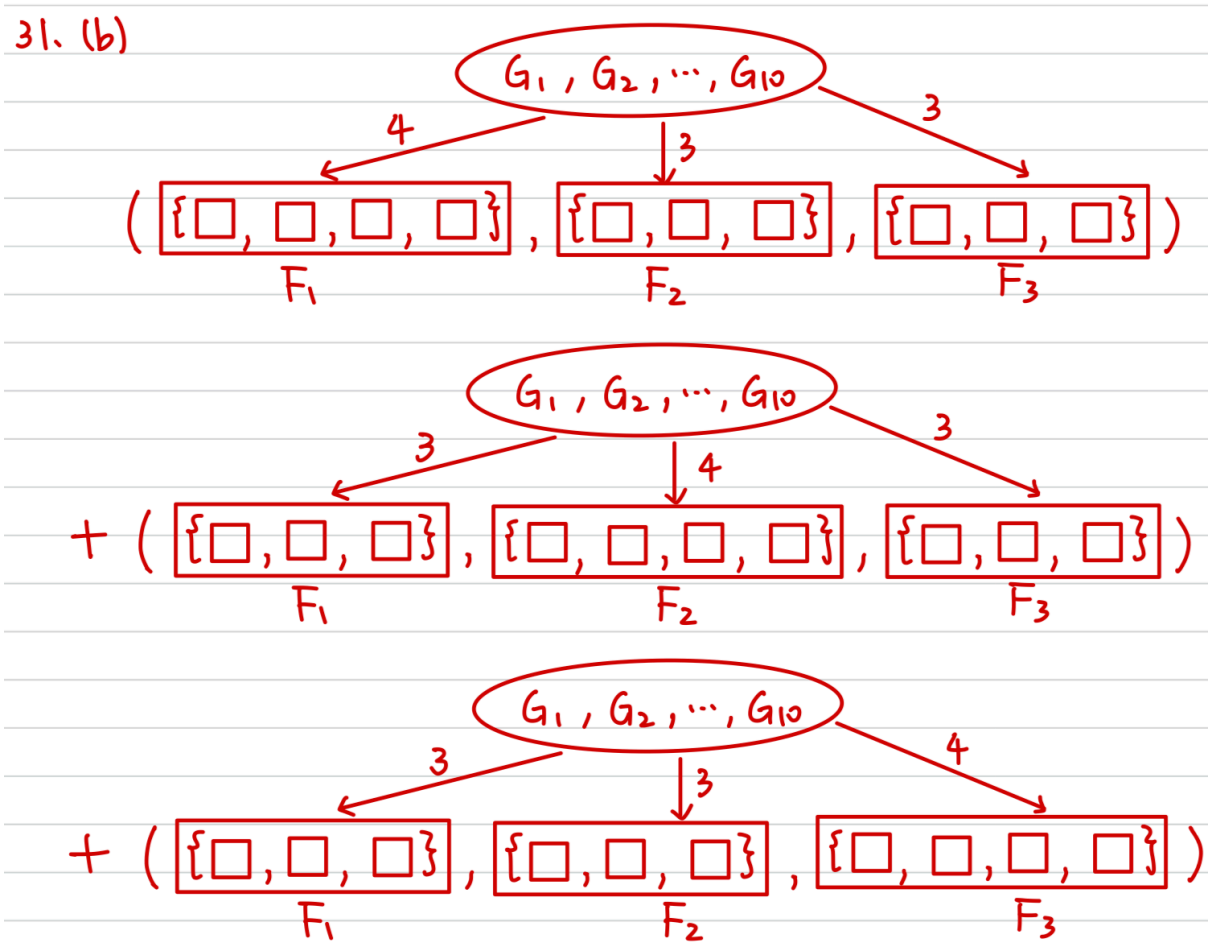
Note that because duplicate combinations will be counted (it's because the remaining 2 friends each receive exactly 3 gifts, so swapping their 2 sets of 3 gifts result in the same combination), you need to multiply by 1/2.

The number of combinations is

$$\binom{6}{3} \times 2 \times \binom{3}{3} \times 1 \times \frac{1}{2}$$

The final answer is

$$\binom{10}{4} \times 3 \times \binom{6}{3} \times 2 \times \binom{3}{3} \times 1 \times \frac{1}{2} = 12600$$



Problem 37.

37.(a)

Let T_i be the i-th type of fish, x_i be the number of T_i , $i = 1, 2, 3, 4, 5$, then our goal is to find the distribution of 10 fish among 5 different types of fish, which is equivalent to finding the number of non-negative integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 10$. The answer is

$$\binom{10+5-1}{5-1} = \binom{14}{4} = 1001$$

37.(b)

Continuing from (a), suppose T_5 represents trout, and T_1, T_2, T_3, T_4 represent the other 4 types of fish. It is known that 3 of these fish are trout, and the remaining 7 fish need to be distributed among the other 4 types, which is equivalent to finding the number of non-negative integer solutions for $x_1 + x_2 + x_3 + x_4 = 7$. The answer is

$$\binom{7+4-1}{4-1} = \binom{10}{3} = 120$$

37.(c)

Continuing from (b), suppose x_5 represents the number of trout. Given that at least 2 of them are trout, we can let $x_5' = x_5 - 2$, where x_5' represents the number of trout other than the 2 trout. Then, $x_5' \geq 0$. Therefore, we need to distribute the remaining 8 fish among the 5 types of fish, which is equivalent to finding the number of non-negative integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5' = 8$. The answer is

$$\binom{8+5-1}{5-1} = \binom{12}{4} = 495$$

Theoretical Exercise 8.

Consider a group consisting of n men and m women. We want to form a group of r people. Then we have two methods to achieve this.

Method 1

In Method 1, we directly use all the people to draw the results.

There are a total of $(n+m)$ people, therefore the number of ways to select r people is

$$\binom{n+m}{r}$$

Method 2

In Method 2, we sample separately for men and women.

Suppose that among the r people we select, there are k men and $(r-k)$ women, then the number of combinations will be

$$\binom{n}{k}\binom{m}{r-k}$$

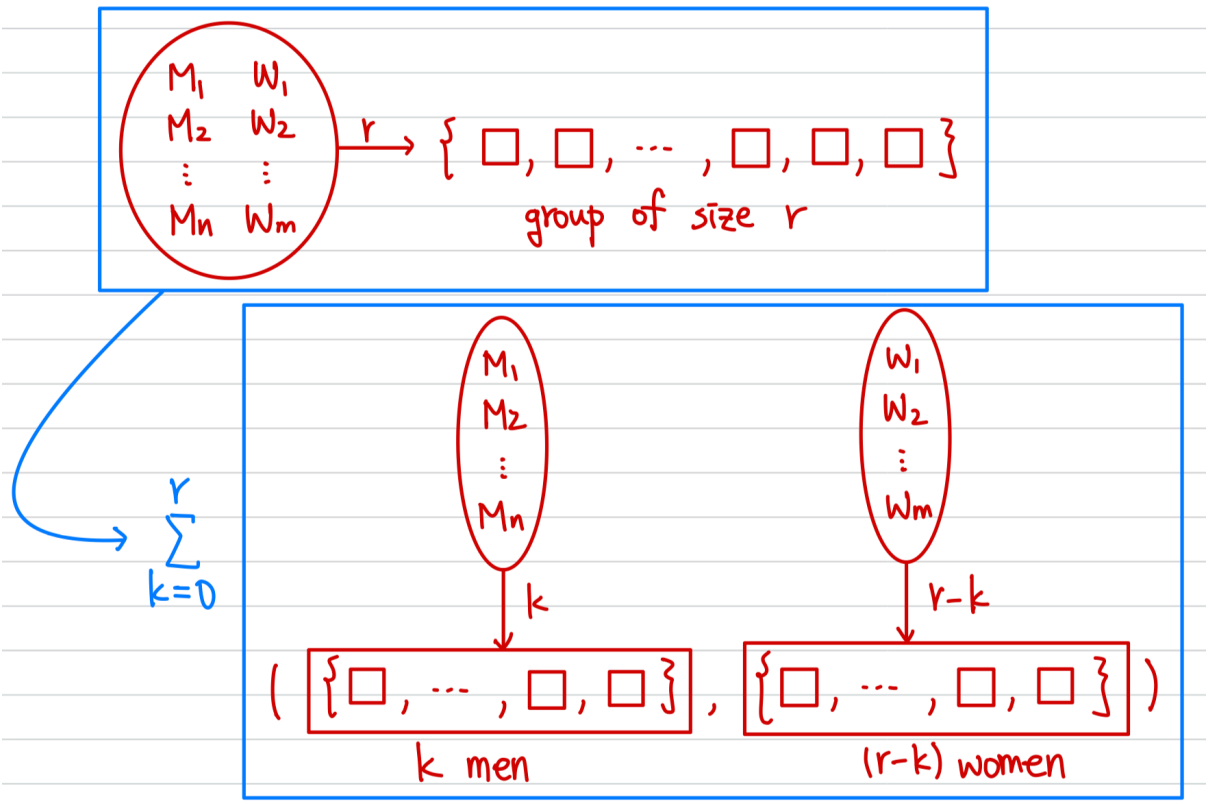
Since k can range from 0 to r, the total number of combinations is

$$\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

Conclusion

Since both of the above methods calculate the number of ways to choose r people from n men and m women. Therefore, the original equation holds true, that is

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$


M1 W1
M2 W2
Mn Wm
r
group of size r
r
k=0
k
r-k
k men
(r-k) women