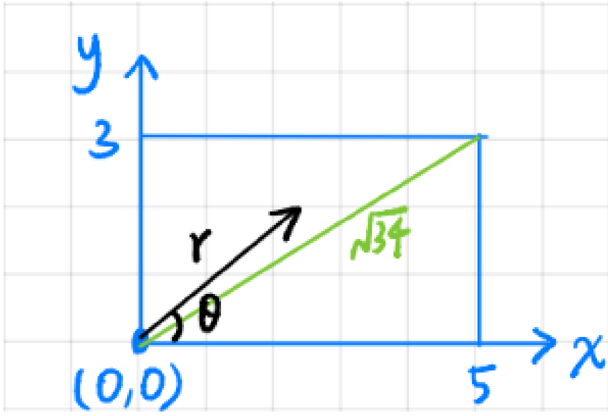
7.5

已知：



令(X,Y)為rescue call 發生的位置，其分配為在上圖矩形上的uniform distribution 。故其聯合機率密度函數為：

$$f(x,y) = \frac{1}{5}\cdot\frac{1}{3} = \frac{1}{15}, \quad 0 \le x \le 5,\ 0 \le y \le 3$$

求期望值：

$$E[\sqrt{X^2+Y^2}] = \int_0^5 \int_0^3 \sqrt{x^2+y^2}\cdot\frac{1}{15}\,dx\,dy$$

解答

改用極座標

令：

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right),$$

$$dx\,dy = r\,dr\,d\theta$$

則期望值轉換為(積分區域請參見上圖圖示)：

$$E[\sqrt{X^2+Y^2}] = \frac{1}{15}\int_0^{\tan^{-1}(\frac{3}{5})}\int_0^{5\sec(\theta)} r^2\,drd\theta \; + \frac{1}{15}\int_{\tan^{-1}(\frac{3}{5})}^{\frac{\pi}{2}}\int_0^{3\csc(\theta)} r^2\,drd\theta$$

$$= \frac{1}{15}\left[\int_0^{\tan^{-1}(\frac{3}{5})} \frac{5^3}{3}\sec^3\theta\,d\theta + \int_{\tan^{-1}(\frac{3}{5})}^{\frac{\pi}{2}} 9\csc^3\theta\,d\theta\right]$$

進一步計算：

$$E[\sqrt{X^2+Y^2}] = \frac{1}{15}\left\{\frac{5^3}{3}\cdot\frac{1}{2}\left[\ln\left(\frac{\sqrt{34}}{5}+\frac{3}{5}\right)+\frac{\sqrt{34}}{5}\,\frac{3}{5}\right]+9\left[\frac{1}{2}\ln\left(\frac{\sqrt{34}}{3}+\frac{5}{3}\right)+\frac{\sqrt{34}}{3}\,\frac{5}{3}\right]\right\}$$

附註(i)

計算$\int \sec^3 x\,dx$：

$$\int \sec^3 x\,dx = \int \sec x\left(1+\tan^2 x\right)\,dx = \int \sec x\,dx + \int \sec x \tan^2 x\,dx$$

1. 第一項：
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C_1$$

2. 第二項透過分部積分計算：令：

$$u = \tan x, \quad dv = \sec x \tan x\,dx$$

則：

$$v = \sec x, \quad du = \sec^2 x\,dx$$

分部積分公式：

$$\int \sec x \tan^2 x\,dx = \sec x \tan x - \int \sec^3 x\,dx$$

代回原公式：

$$\int \sec^3 x\,dx = \ln|\sec x + \tan x| + \sec x \tan x - \int \sec^3 x\,dx + C_1$$

整理後得到：

$$2\int \sec^3 x\,dx = \ln|\sec x + \tan x| + \sec x \tan x + C_1$$

最終結果爲：

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\ln|\sec x + \tan x| + \sec x \tan x\right) + C_2$$

附註(ii)

計算$\int \csc^3 x\,dx$：

$$\int \csc^3 x\,dx = \int \csc x\left(1+\cot^2 x\right)\,dx = \int \csc x\,dx + \int \csc x \cot^2 x\,dx$$

1. 第一項：
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C_3$$

2. 第二項透過分部積分計算：令：

$$u = \cot x, \quad dv = \csc x \cot x\,dx$$

則：

$$v = -\csc x, \quad du = -\csc^2 x\, dx$$

分部積分公式：

$$\int \csc x \cot^2 x\, dx = -\csc x \cot x - \int \csc^3 x\, dx$$

代回原公式：

$$\int \csc^3 x\, dx = -\ln|\csc x + \cot x| - \csc x \cot x - \int \csc^3 x\, dx + C_3$$

整理後得到：

$$2\int \csc^3 x\, dx = -\ln|\csc x + \cot x| - \csc x \cot x + C_3$$

最終結果爲：

$$\int \csc^3 x\, dx = -\frac{1}{2}\left(\ln|\csc x + \cot x| + \csc x \cot x\right) + C_4$$

7.9

共有n 個球，編號爲1 到n，放入n 個瓶子，這些瓶子也編號爲1 到n。其中球i只能放入瓶1,2,...,i。求：

1. (a) 空瓶子的期望數量；
2. (b) 沒有瓶子是空的概率。

解答

(a) 空瓶子的期望數量

定義：

$$X_i = \begin{cases} 1, & \text{如果瓶子} i \text{ 是空的}, \\ 0, & \text{否則}. \end{cases}$$

其中$i = 1, \ldots, n$。令：

$$X = \sum_{i=1}^{n} X_i$$

則X 爲空瓶的數量。
計算期望值：

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} P(X_i = 1)$$

概率$P(X_i = 1)$ 表示瓶子i 是空的，也就是所有球都不在瓶子i 中：

$$\begin{aligned} P(X_i = 1) &= \sum_{i=1}^{w} P(\text{ball 1 not in urn } i, \ldots, \text{ball } i \text{ not in urn } i, \ldots, \text{ball } n \text{ not in urn } i) \\ &= \sum_{i=1}^{n} \left(\frac{i-1}{i} \cdot \frac{i}{i+1} \cdots \frac{n-1}{n}\right) \\ &= \frac{i-1}{n}. \end{aligned}$$

因此：

$$E(X) = \sum_{i=1}^{n} \frac{i-1}{n} = \frac{n-1}{2}$$

(b) 沒有瓶子是空的概率

求$P(X=0)$，即沒有瓶子是空的概率：

$$P(X=0) = P(\text{每個瓶子都有一個球}) = P(\text{第i個球在第i個瓶子中}, i=1,\ldots,n)$$

$$\begin{aligned} P(X=0) &= P(\text{ball 1 in urn 1}, \text{ball 2 in urn 2}, \ldots, \text{ball } n \text{ in urn } n) \\ &= 1 \cdot \frac{1}{2} \cdot \cdots \cdot \frac{1}{n} = \frac{1}{n!}. \end{aligned}$$

7.21

(a) 包含正好2 個人的房間的期望數量

令X 表示包含正好2 個人的房間數量。對於$i = 1, \ldots, 200$，定義指示變數：

$$X_i = \begin{cases} 1, & \text{如果房間} i \text{ 包含正好2 個人,} \\ 0, & \text{否則.} \end{cases}$$

則根據距二項分布的性質，

$$X_i \text{ 的marginal distribution 爲Bernoulli}(p = \binom{50}{2} \left(\frac{1}{200}\right)^2 \left(1 - \frac{1}{200}\right)^{48})$$

並且

$$X = \sum_{i=1}^{200} X_i$$

計算期望值：

$$E(X) = E\left(\sum_{i=1}^{200} X_i\right) = \sum_{i=1}^{200} E(X_i) = \sum_{i=1}^{200} P(X_i = 1) = 200p = 200 \cdot \binom{50}{2} \left(\frac{1}{200}\right)^2 \left(1 - \frac{1}{200}\right)^{48}$$

注意在此計算中，我們只需要使用到$X_1, \ldots, X_{200}$ 的marginal distribution。無須使用到它們的joint distribution，這使得計算能簡化。

(b) 至少包含1 個人的房間的期望數量

令Y 表示非空房間的數量。對於$i = 1, \ldots, 200$，定義指示變數：

$$Y_i = \begin{cases} 1, & \text{如果房間} i \text{ 不空}, \\ 0, & \text{否則}. \end{cases}$$

則Y_i的marginal distribution 爲Bernoulli(q)，其中：

$$q = P(Y_i = 1) = 1 - P(\text{房間} i \text{ 是空的}) = 1 - \left(1 - \frac{1}{200}\right)^{50}$$

並且

$$Y = \sum_{i=1}^{200} Y_i$$

計算期望值：

$$E(Y) = E\left(\sum_{i=1}^{200} Y_i\right) = \sum_{i=1}^{200} E(Y_i) = \sum_{i=1}^{200} P(Y_i = 1) = 200q = 200 \cdot \left[1 - \left(1 - \frac{1}{200}\right)^{50}\right]$$

注意在此計算中，我們只需要使用到$Y_1, \ldots, Y_{200}$ 的marginal distribution。無須使用到它們的joint distribution，這使得計算能簡化。

7.41

考慮：

$$\text{Cov}(Y_n, Y_{n+j}) = \text{Cov}\left(\sum_{i=0}^{2} X_{n+i}, \sum_{k=0}^{2} X_{n+j+k}\right)$$

根據協方差的線性性質展開：

$$\text{Cov}(Y_n, Y_{n+j}) = \sum_{i=0}^{2}\sum_{k=0}^{2} \text{Cov}(X_{n+i}, X_{n+j+k})$$

由於X_i 是相互獨立且具有相同的期望值μ 和方差σ^2，我們有：

$$\text{Cov}(X_{i_1}, X_{i_2}) = \begin{cases} \sigma^2, & \text{如果} i_1 = i_2, \\ 0, & \text{如果} i_1 \neq i_2. \end{cases}$$

因此，協方差$\text{Cov}(Y_n, Y_{n+j})$ 的計算結果爲：

$$\text{Cov}(Y_n, Y_{n+j}) = \begin{cases} 3\sigma^2, & j = 0, \\ 2\sigma^2, & j = 1, \\ \sigma^2, & j = 2, \\ 0, & j \geq 3. \end{cases}$$

7.42

Let $f(x,y) = \begin{cases} \frac{1}{y}e^{-(y+\frac{x}{y})}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

Then, the marginal pdf of Y is

$$f_Y(y) = \int_0^\infty \frac{1}{y}e^{-(y+\frac{x}{y})}dx = \frac{e^{-y}}{y}\int_0^\infty e^{-\frac{x}{y}}dx = e^{-y} \quad for \quad 0 \le y \le \infty$$

So, $Y \sim exp(1)$, which implies that E(Y)=1.

Since $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{y}e^{-\frac{x}{y}} \quad for \quad x>0 \implies \quad X|Y=y \sim exp(\frac{1}{y}) \implies E(X|Y) = \frac{1}{1/Y} = Y$.

By the Law of total expectation,

$$E(X) = E(E(X|Y)) = E(Y) = 1 \ ,$$

$$E(XY) = E(E(XY|Y)) = E(YE(X|Y)) = E(Y^2) = Var(Y) + [E(Y)]^2 = 2 \ ,$$

where $X|Y=y \sim exp(\frac{1}{y})$.

Note that

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 2-1 = 1 \ .$$

Alternatively, if not using conditional expectation , E(X), E(Y), and E(XY) can also be calculated by double integration as shown below:

$$E(X) = \int_0^\infty \int_0^\infty xf(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty \frac{x}{y}e^{-(y+\frac{x}{y})}\,dx\,dy = \int_0^\infty \frac{e^{-y}}{y}\int_0^\infty xe^{-\frac{x}{y}}\,dx\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\left[-xye^{-\frac{x}{y}}\Big|_0^\infty + y\int_0^\infty e^{-\frac{x}{y}}\,dx\right]dy = \int_0^\infty \frac{e^{-y}}{y}\left[y\int_0^\infty e^{-\frac{x}{y}}\,dx\right]dy$$

$$= \int_0^\infty e^{-y}\int_0^\infty e^{-\frac{x}{y}}\,dx\,dy = \int_0^\infty ye^{-y}\,dy = -ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y}\,dy = 1$$

$$E(Y) = \int_0^\infty \int_0^\infty yf(x,y)dxdy = \int_0^\infty \int_0^\infty e^{-(y+\frac{x}{y})}dxdy = \int_0^\infty e^{-y}\int_0^\infty e^{-\frac{x}{y}}dxdy$$

$$= \int_0^\infty ye^{-y}dy = -ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y}\,dy = 1$$

$$E(XY) = \int_0^\infty \int_0^\infty xyf(x,y)dxdy = \int_0^\infty \int_0^\infty xe^{-(y+\frac{x}{y})}dxdy = \int_0^\infty e^{-y} \int_0^\infty xe^{-\frac{x}{y}}dxdy$$
$$= \int_0^\infty e^{-y} \left[-xye^{-\frac{x}{y}} \Big|_0^\infty + y\int_0^\infty e^{-\frac{x}{y}}\,dx \right] dy = \int_0^\infty y^2e^{-y}dy = -y^2e^{-y}\Big|_0^\infty + 2\int_0^\infty ye^{-y}dy = 2$$

Hence, Cov(X,Y)=E(XY)-E(X)E(Y)=1 .

7.47

By question, $X_i \sim \Gamma(k,\theta)$, $i=1,2,3$, and $Cov(X_i,X_j)=0\ \ \forall\ i\neq j$.

(a)

Since
$$Cov(X_1, X_1+X_2) = Var(X_1) + Cov(X_1,X_2) = \frac{k}{\theta^2},$$

and $Var(X_1+X_2) = Var(X_1)+Var(X_2)+2Cov(X_1,X_2) = \frac{2k}{\theta^2}$.

So,
$$Corr(X_1, X_1+X_2) = \frac{Cov(X_1,X_1+X_2)}{\sqrt{var(X_1)var(X_1+X_2)}} = \frac{\frac{k}{\theta^2}}{\sqrt{\frac{k}{\theta^2}\times 2\frac{k}{\theta^2}}} = \frac{\sqrt{2}}{2}$$

Note that the mean and variance of $\Gamma(k,\theta)$ are $\frac{k}{\theta}$ and $\frac{k}{\theta^2}$, respectively.

(b)

Since
$$Cov(X_1+2X_2, X_1+X_2+X_3) = Var(X_1) + 2Var(X_2) = \frac{3k}{\theta^2}$$

and since X_1, X_2, X_3 are pairwise uncorrelated , $Var(X_1+2X_2) = Var(X_1)+4Var(X_2) = \frac{5k}{\theta^2}$, $Var(X_1+X_2+X_3) = Var(X_1)+Var(X_2)+Var(X_3) = \frac{3k}{\theta^2}$.
So,

$$Corr(X_1+2X_2, X_1+X_2+X_3) = \frac{Cov(X_1+2X_2,X_1+X_2+X_3)}{\sqrt{var(X_1+2X_2)var(X_1+X_2+X_3)}} = \frac{\frac{3k}{\theta^2}}{\sqrt{\frac{5k}{\theta^2}\times\frac{3k}{\theta^2}}} = \frac{\sqrt{15}}{5}$$

7.49

(a)

By question, D_i is the number of edges that have vertex i as one of their vertices.
如果第i個頂點有和其他(n-1)個頂點連到線算成功(連線成功機率爲p)，若沒有連到線則算失敗(失敗機率爲1-p)，且每個頂點都是獨立連線$\Longrightarrow$ independent Bernoulli
由於D_i爲第i個頂點和其他共(n-1)個頂點的總連線成功次數，因此總成功次數
$D_i \sim Bin(n-1,p)$, for i=1,...,n. $\Longrightarrow \quad Var(D_i) = (n-1)p(1-p)$

(b)

For $k < l\ ,\ k,l = 1,...,n.$

let $E_{k,l} = \begin{cases} 1, & \text{if there is an edge between vertex k and vertex l} \\ 0, & \text{otherwise} \end{cases}$

Clearly, $E_{k,l} \sim Ber(p)\ ,\ and\ \ E_{k,l}\ 's\ \ are\ \ independent.$
And then, we have

$$D_i = \sum_{k<i} E_{k,i} + \sum_{l>i} E_{i,l}$$
$$D_j = \sum_{k<j} E_{k,j} + \sum_{l>j} E_{j,l}$$

So, for $i < j$,

$$\begin{aligned} Cov(D_i, D_j) &= Cov(\sum_{k<i} E_{k,i} + \sum_{l>i} E_{i,l}\ , \sum_{k<j} E_{k,j} + \sum_{l>j} E_{j,l}) \\ &= Cov(\sum_{k<i} E_{k,i} + \sum_{l>i, l\neq j} E_{i,l} + E_{i,j}\ ,\ \sum_{k<j, k\neq i} E_{k,j} + \sum_{l>j} E_{j,l} + E_{i,j}) \\ &= Cov(E_{i,j}\ ,\ E_{i,j}) = Var(E_{i,j}) \\ &= p(1-p) \end{aligned}$$

Hence,

$$Corr(D_i, D_j) = \frac{Cov(D_i, D_j)}{\sqrt{Var(D_i)Var(D_j)}} = \frac{p(1-p)}{\sqrt{(n-1)p(1-p) \times (n-1)p(1-p)}} = \frac{1}{n-1}$$

$$\Longrightarrow\ Corr(D_i, D_j) = \begin{cases} 1\ , & if\ i = j \\ \frac{1}{n-1}, & \forall\ i \neq\ j \end{cases}$$

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7.53

The marginal pdf of X is

$$f_X(x) = \begin{cases} \int_0^x 10x^2y\ dy = 10x^2[\frac{y^2}{2}]|_0^x = 5x^4\ , & for\ 0 < x < 1 \\ 0\ ,\ otherwise \end{cases}$$

Then , $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{10x^2y}{5x^4} = \frac{2y}{x^2}$, $for\ 0 < y < x$, $0 < x < 1$.
So, for $0 < x < 1$,

$$\begin{aligned} E(Y^k|X = x) &= \int_0^x y^k f_{Y|X}(y|x)dy \\ &= 2\int_0^x \frac{y^{k+1}}{x^2}dy \\ &= \frac{2}{x^2}\left[\frac{y^{k+2}}{k+2}\right]\Big|_0^x \\ &= \frac{2x^k}{k+2} \end{aligned}$$

7.57

- Let N=number of ducks in a flock.Then N$\sim$ $Poisson(6)$.
- 10 hunters target their ducks independently
- 10 hunters fire with hit rate 0.6
- For $1 \le i \le N$,let $X_i = \begin{cases} 1, & i-th\ duck\ be\ hit, \\ 0, & i-th\ duck\ not\ be\ hit. \end{cases}$

  Then X=$\sum_{i=1}^{N} X_i$ is total number of ducks being hit.

Given $N=n$, $X_i|N=n \sim Bernoulli(p)$,

where $p = P(X_i = 1|N=n) = 1 - P(no\ hunter\ hit\ i\text{-}th\ duck)$

$$= \quad 1 - \prod_{h=1}^{10} \underbrace{(\frac{1}{n}\cdot 0.4 + \frac{n-1}{n}\cdot 1)}_{\substack{\text{第 h 個獵人選到i-th duck} \\ \text{但沒射到，或是沒選到i-th duck}}} = 1 - (\tfrac{0.4}{n} + \tfrac{n-1}{n})^{10} = 1 - (1 - \tfrac{0.6}{n})^{10}$$

So, given N=n,

$$\begin{aligned} E(X|N=n) &= E(\sum_{i=1}^{n} X_i|N=n) \\ &= \sum_{i=1}^{n} E(X_i|N=n) \\ &= \sum_{i=1}^{n} (1 \times P(X_i=1|N=n) + 0 \times P(X_i=0|N=n)) \\ &= \sum_{i=1}^{n} (1 \times p + 0) = n \cdot (1 - (1 - \frac{0.6}{n})^{10}) \end{aligned}$$

Thus,
ANS:

$$\begin{aligned} E(X) &= E_N(E(X|N)) \\ &= E_N(N \cdot (1 - (1 - \frac{0.6}{N})^{10})) \\ &= \sum_{n=1}^{\infty} n \cdot (1 - (1 - \frac{0.6}{n})^{10}) \cdot P(N=n) \\ &= \underline{\sum_{n=1}^{\infty} n \cdot (1 - (1 - \frac{0.6}{n})^{10}) \cdot \frac{e^{-6}6^n}{n!}} \quad \# \end{aligned}$$

** 注意在此計算中，我們只需使用到 給定 N=n 時，$X_1, X_2 ..., X_n$ 的 marginal distribution ，不須用到它的 joint distribution，使得計算可以簡化。

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(在給定 N=n 時，$X_1, X_2, ..., X_n$ 並非 independent，故其 joint distribution 無法由其 marginal distribution 唯一決定。而給定 N=n 時，X=$\sum X_i$ 的分配也不是 binomial(n,p))

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7.70

Let X be number of storms next year.

Let $Y = \begin{cases} 1, & \textit{next year is good year} \\ 0, & \textit{next year is bad year} \end{cases} \Rightarrow Y \sim Ber(0.4)$

$$\Rightarrow X|Y=1 \sim Poi(3)$$
$$X|Y=0 \sim Poi(5)$$

$$\therefore E(X|Y=1)=3, VAR(X|Y=1)=3$$
$$E(X|Y=0)=5, VAR(X|Y=0)=5$$

Then,

$$\begin{aligned} E(X) &= E_Y(E_{X|Y}(X|Y)) \\ &= E(X|Y=1)\cdot 0.4 + E(X|Y=0)\cdot 0.6 \\ &= 3\cdot 0.4 + 5\cdot 0.6 = 4.2 \end{aligned}$$

Similarly,

$$\begin{aligned} Var(X) &= E_Y(Var_{X|Y}(X|Y)) + Var_Y(E_{X|Y}(X|Y)) \\ &= (0.4\cdot 3 + 0.6\cdot 5) + \left[(0.4\cdot(3-4.2)^2 + 0.6\cdot(5-4.2)^2)\right] \\ &= 4.2 + 0.96 = \underline{5.16}_{\#} \end{aligned}$$

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7.79

$X \sim U(0,1)$, $a_0 = 0$, $a_1 = \frac{1}{2}$, $a_2 = 1$

Let $I = \begin{cases} 0, & if \ \ a_0 = 0 < X \le \frac{1}{2} = a_1 \\ 1, & if \ \ a_1 = \frac{1}{2} < X \le 1 = a_2 \end{cases}$ $\Rightarrow$

$the\ pdf\ of\ X|I=0\ is \begin{cases} \frac{1}{P(0<X\le\frac{1}{2})} = 2, & 0 < x \le \frac{1}{2} \\ 0, & otherwise \end{cases}$

$the\ pdf\ of\ X|I=1\ is \begin{cases} \frac{1}{P(\frac{1}{2}<X\le 1)} = 2, & \frac{1}{2} < x \le 1 \\ 0, & otherwise \end{cases}$

(a)

By proposition 6.1,$y_i = E(X|I=i) = E(X|a_i < X \le a_{i+1})$,$i = 0,1$

Thus,

$$ifI = 0(i.e.\ \ 0 < X \le \frac{1}{2}), y_0 = E(X|I=0) = \int_0^{\frac{1}{2}} 2xdx = \frac{1}{4}$$

$$ifI = 1(i.e.\ \ \frac{1}{2} < X \le 1), y_1 = E(X|I=1) = \int_{\frac{1}{2}}^{1} 2xdx = \frac{3}{4}$$

(b)

$$E(X-Y)^2 = \sum_{i=0}^{1} E((X-y_i)^2|I=i)\cdot P(I=i)$$

$$= \frac{1}{2}\cdot(\int_0^{\frac{1}{2}} 2(x-\frac{1}{4})^2dx + \int_{\frac{1}{2}}^{1} 2(x-\frac{3}{4})^2dx) = \underline{\frac{1}{48}}_{\#}$$

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7.82

$$f_{X,Y}(x,y)=\frac{1}{\sqrt{2\pi}}e^{-y}e^{-\frac{(x-y)^2}{2}}\qquad \begin{array}{l}0<y<\infty\\ -\infty<x<\infty\end{array}$$

(a)

$$\begin{aligned}
M_{X,Y}(t_1,t_2)=E(e^{t_1X+t_2Y})&=\int_0^\infty\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{t_1x+t_2y}e^{-y}e^{-\frac{(x-y)^2}{2}}dxdy\\
&=\int_0^\infty\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{t_1x+t_2y}e^{-y}e^{-\frac{x^2-2xy+y^2}{2}}dxdy\\
&=\int_0^\infty e^{t_2y-y-\frac{y^2}{2}}\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{t_1x-\frac{x^2}{2}+xy}dxdy\\
&=\int_0^\infty e^{t_2y-y-\frac{y^2}{2}}\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\cdot(x^2-2(t_1+y)x+(t_1+y)^2-(t_1+y)^2)}dxdy\\
&=\int_0^\infty e^{t_2y-y-\frac{y^2}{2}}e^{(t_1+y)^2/2}\underbrace{\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\cdot(x-(t_1+y))^2}dx}_{pdf\ of\ N(t_1+y,1)\Rightarrow\text{積分}=1}\,dy\\
&=e^{\frac{t_1^2}{2}}\int_0^\infty e^{t_2y-y+t_1y}dy=e^{\frac{t_1^2}{2}}\int_0^\infty e^{(t_2+t_1-1)y}dy\\
&=\begin{cases}e^{\frac{t_1^2}{2}}\frac{1}{1-t_1-t_2} & ,if\ \ t_1+t_2<1\\ do\ not\ exist & ,if\ \ t_1+t_2\geq 1\end{cases}
\end{aligned}$$

(b)

$$M_X(t_1)=M_{X,Y}(t_1,0)=e^{\frac{t_1^2}{2}}\frac{1}{1-t_1}\quad if\ t_1<1$$

$$M_Y(t_2)=M_{X,Y}(0,t_2)=1\cdot\frac{1}{1-t_2}\quad if\ t_2<1$$

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TE6

For $i = 1, ..., n$, let $X_i = I\{A_i \text{ occurs}\} = \begin{cases} 1 & \text{if A}_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases} \Rightarrow X_i \sim Bernoulli(p_i = P(A_i))$ and $X = \sum_{i=1}^{n} X_i$

RHS of equation：

$$\begin{aligned}\sum_{k=1}^{n} P(A_k) &= \sum_{k=1}^{n} E[X_k] \\ &= E[X]\end{aligned}$$

LHS of equation：

$$\begin{aligned}\sum_{k=1}^{n} P(C_k) &= \sum_{k=1}^{n} P(X \geq k) \\ &= \sum_{k=1}^{n} \sum_{s=k}^{n} P(X = s) \\ &= \sum_{s=1}^{n} P(X = s) + \sum_{s=2}^{n} P(X = s) + \cdots + \sum_{s=n}^{n} P(X = s) \\ &= \sum_{s=1}^{n} s \cdot P(X = s) \\ &= \sum_{s=0}^{n} s \cdot P(X = s) \\ &= E[X]\end{aligned}$$

提醒：LHS 其實就是Homework 4中Theoretical EX.5中的推導。

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TE18

$$\begin{aligned}
\hat{\mu} &= \lambda X_1 + (1-\lambda)X_2 \\
E[\hat{\mu}] &= \lambda\mu + (1-\lambda)\mu = \mu \\
\because X_1 \& X_2 \ indep. \Rightarrow Var(\hat{\mu}) &= \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2 \\
&= \lambda^2(\sigma_1^2+\sigma_2^2) - 2\lambda\sigma_2^2 + \sigma_2^2 \\
&= (\sigma_1^2+\sigma_2^2)(\lambda - \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2})^2 - \frac{\sigma_2^4}{\sigma_1^2+\sigma_2^2} + \sigma_2^2
\end{aligned}$$

$\therefore \underset{\lambda}{argmin} Var(\hat{\mu}) = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$
$\because E[\hat{\mu}] = \mu \therefore Var(\hat{\mu}) = E\{(\hat{\mu}-\mu)^2\} =$ mean square error(MSE) of $\hat{\mu}$ for estimating μ,其中$\hat{\mu}-\mu$為估計誤差。
因此，找$Var(\hat{\mu})$ 的最小值等價於找最小的MSE。

TE52

$X_1, ..., X_n$ are independent and identically distributed exponential random variables, each having mean $\frac{1}{\lambda}$.
By Table 7.2, we can know the mgf of X_i is $M_{X_i}(t) = \frac{\lambda}{\lambda-t}$.
Let $Y = \sum_{i=1}^n X_i$, the mgf of Y,

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) \\
&= E(e^{t\sum_{i=1}^n X_i}) \\
&= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_n}) \\
&= \prod_{i=1}^n M_{X_i}(t) \\
&= (\frac{\lambda}{\lambda-t})^n \ \leftarrow \text{由Table7.2得知，這是Gamma(n,}\lambda\text{)的mgf}
\end{aligned}$$

Hence, by the uniqueness of mgf, $Y = \sum_{i=1}^n X_i \sim Gamma(n, \lambda)$