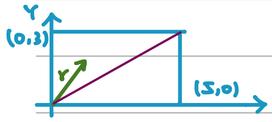
7.5



$f(x,y)=\frac{1}{5}\cdot\frac{1}{3}=\frac{1}{15},\ 0\le x\le 5,\ 0\le y\le 3 \Rightarrow E(R)=E(\sqrt{x^2+y^2})=\int_0^5\int_0^3\sqrt{x^2+y^2}\cdot\frac{1}{15}\,dx\,dy$

Let $r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}(\frac{y}{x}) \Rightarrow E(\sqrt{x^2+y^2})=\frac{1}{15}\int_0^{\tan^{-1}(\frac{3}{5})}\int_0^{5\sec(\theta)} r^2\,dr\,d\theta+\frac{1}{15}\int_{\tan^{-1}(\frac{3}{5})}^{\pi/2}\int_0^{3\csc(\theta)} r^2\,dr\,d\theta$

$=\frac{1}{15}\left(\int_0^{\tan^{-1}(3/5)}\frac{5^3}{3}\sec^3\theta\,d\theta+\int_{\tan^{-1}(3/5)}^{\frac{\pi}{2}}9\csc^3\theta\,d\theta\right)=\frac{1}{15}\left(\frac{5^3}{3}\cdot\frac{1}{2}\left(\ln\left(\frac{\sqrt{34}}{5}+\frac{3}{5}\right)+\frac{\sqrt{34}}{5}\cdot\frac{3}{5}\right)+9\left(\frac{1}{2}\ln\left(\frac{\sqrt{34}}{3}+\frac{5}{3}\right)+\frac{\sqrt{34}}{3}\frac{5}{3}\right)\right)$

補充

$$\int \sec^3 x\,dx=\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln|\sec x+\tan x|+C$$

最短推導想法

1. 拆成：
$\sec^3 x=\sec x(\sec^2 x)$
2. 看到 $\sec^2 x$ → 就想到 $d(\tan x)=\sec^2 x\,dx$
3. 分部積分：

令

$u=\sec x,\quad dv=\sec^2 x dx$

$du=\sec x\tan x dx,\quad v=\tan x$

帶入：

$\int \sec^3 x dx=\sec x\tan x-\int \tan x\sec x\tan x dx$

再用

$\tan^2 x=\sec^2 x-1$

整理後便得公式。

$$\int \csc^3 x\,dx=-\frac{1}{2}\csc x\cot x+\frac{1}{2}\ln|\csc x-\cot x|+C$$

最短推導想法

1. 拆成：
$\csc^3 x=\csc x(\csc^2 x)$
2. 看到 $\csc^2 x$ → 就想到 $d(\cot x)=-\csc^2 x\,dx$
3. 分部積分：

令

$u=\csc x,\quad dv=\csc^2 x dx$

$du=-\csc x\cot x dx,\quad v=-\cot x$

整理後得到公式。

7.9

(a) Let $X_i=\begin{cases}1; & \text{if } i\text{th urn is empty}\\ 0; & \text{others}\end{cases}$, Let X be the # of urns that are empty, Then $X=\sum X_i$

$E(X)=E(\sum_{i=1}^n X_i)=\sum_{i=1}^n E(X_i)=\sum_{i=1}^n P(X_i=1)$

and $P(X_i=1)=P(\text{ball 1 not in urn } i,\ \text{ball 2 not in urn } i,\cdots,\ \text{ball } n \text{ not in urn } i)$

$=P(\text{ball } i \text{ not in urn } i,\cdots,\text{ball } n \text{ not in urn } i)=\frac{i-1}{i}\cdot\frac{i}{i+1}\cdots\cdot\frac{n-1}{n}=\frac{i-1}{n}$

$\Rightarrow E(X)=\sum_{i=1}^n P(X_i=1)=\sum_{i=1}^n\frac{i-1}{n}=\frac{n-1}{2}.$

(b) $P(X=0)=P(\text{each urn has a ball})=P(i^{th}\text{ ball in the } i^{th}\text{ urn},\ 1\le i\le n)$

$=1\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\cdots\frac{1}{n}=\frac{1}{n!}.$

Problem 7.21

Problem. Fifty people are placed randomly in 200 rooms. Compute (a) the expected number of rooms containing exactly 2 people; (b) the expected number of non-vacant rooms.

We assume each of the 50 people independently chooses one of the 200 rooms with equal probability 1/200.

Let N_j be the number of people in room j, $j = 1, \ldots, 200$. Then, for each fixed j,

$$N_j \sim \text{Binomial}\left(50, \frac{1}{200}\right),$$

since each person independently goes to room j with probability 1/200.

(Note

$$(N_1, \ldots, N_{200}) \sim \text{Multinomial}\left(50, 200, \frac{1}{200}, \ldots, \frac{1}{200}\right),$$

但在此題中不必使用它們的 joint distribution，只須用到它們的 marginal distributions.)

(a) Expected number of rooms with exactly two people

For each room $j = 1, 2, \ldots, 200$, define the indicator

$$I_j = \begin{cases} 1, & \text{if room } j \text{ contains exactly 2 people}, \ (i.e. N_j = 2) \\ 0, & \text{otherwise.} (i.e. N_j \neq 2) \end{cases}$$

Then the total number of rooms with exactly 2 people T is

$$T = \sum_{j=1}^{200} I_j.$$

By linearity of expectation,

$$E[T] = \sum_{j=1}^{200} E[I_j].$$

Since all rooms are identically distributed,

$$E[I_j] = P(N_j = 2).$$

Because $N_j \sim \text{Binomial}(50, 1/200)$,

$$P(N_j = 2) = \binom{50}{2}\left(\frac{1}{200}\right)^2\left(1 - \frac{1}{200}\right)^{50-2} = \binom{50}{2}\left(\frac{1}{200}\right)^2\left(\frac{199}{200}\right)^{48}.$$

Therefore,

$$E[T] = 200 \cdot \binom{50}{2}\left(\frac{1}{200}\right)^2\left(\frac{199}{200}\right)^{48}.$$

made by 侯秉逸, 張劭謙, 呂心樂, 徐瑋崙, 李友棣, 黃子維 助教

So the expected number of rooms containing exactly two people is

$$\boxed{E[\#\text{ rooms with exactly 2 people}] = 200\binom{50}{2}\left(\frac{1}{200}\right)^2\left(\frac{199}{200}\right)^{48}}.$$

注意在此題的計算中，只須使用到 $(I_1, \ldots, I_{200})$ 的 marginal distributions，無須使用到它們的 joint distribution。這使得計算能簡化。

但有興趣的同學，可嘗試推導它們的 joint distribution。

(注意： $I_1, \ldots, I_{200}$ are not independent because they are transformations of $N_1, \ldots, N_{200}$, respectively, but $N_1, \ldots, N_{200}$ are not independent.**)**

(b) Expected number of non-vacant rooms

Now define, for each room $j = 1, 2, \ldots, 200$,

$$J_j = \begin{cases} 1, & \text{if room } j \text{ is non-vacant (i.e. has at least one person,} N_j \geq 1) \\ 0, & \text{if room } j \text{ is vacant.} (i.e. N_j = 0) \end{cases}$$

Then the total number of non-vacant rooms S is

$$S = \sum_{j=1}^{200} J_j,$$

and again by linearity of expectation,

$$E[S] = \sum_{j=1}^{200} E[J_j].$$

For a fixed room j,

$$E[J_j] = P(N_j \geq 1) = 1 - P(N_j = 0).$$

But

$$P(N_j = 0) = \left(1 - \frac{1}{200}\right)^{50} = \left(\frac{199}{200}\right)^{50}.$$

Hence

$$E[J_j] = 1 - \left(\frac{199}{200}\right)^{50}.$$

Therefore,

$$E[S] = 200\left[1 - \left(\frac{199}{200}\right)^{50}\right].$$

So the expected number of non-vacant rooms is

$$\boxed{E[\#\text{ non-vacant rooms}] = 200\left[1 - \left(\frac{199}{200}\right)^{50}\right]}.$$

注意在此題的計算中，只須使用到 $(J_1, \ldots, J_{200})$ 的 marginal distributions，無須使用到它們的 joint distribution。這使得計算能簡化。

但有興趣的同學，可嘗試推導它們的 joint distribution。

(注意： $J_1, \ldots, J_{200}$ are not independent, because they are transformations of $N_1, \ldots, N_{200}$, respectively, but $N_1, \ldots, N_{200}$ are not independent.**)**

Problem 7.41

Problem. Let $X_1, X_2, \ldots$ be independent with common mean μ and common variance σ^2, and set

$$Y_n = X_n + X_{n+1} + X_{n+2}.$$

For $j \geq 0$, find $\mathrm{Cov}(Y_n, Y_{n+j})$.

Solution. Write

$$Y_n = X_n + X_{n+1} + X_{n+2}, \qquad Y_{n+j} = X_{n+j} + X_{n+j+1} + X_{n+j+2}.$$

Using bilinearity of covariance,

$$\mathrm{Cov}(Y_n, Y_{n+j}) = \mathrm{Cov}\left(\sum_{k=0}^{2} X_{n+k}, \sum_{\ell=0}^{2} X_{n+j+\ell}\right) = \sum_{k=0}^{2}\sum_{\ell=0}^{2} \mathrm{Cov}\left(X_{n+k}, X_{n+j+\ell}\right).$$

Because the X_i are independent, we have

$$\mathrm{Cov}(X_r, X_s) = \begin{cases} \sigma^2, & r = s, \\ 0, & r \neq s. \end{cases}$$

Hence each nonzero term in the double sum corresponds to an index that appears in both Y_n and Y_{n+j}; each such overlap contributes σ^2.

The set of indices in Y_n is $\{n, n+1, n+2\}$, and in Y_{n+j} is $\{n+j, n+j+1, n+j+2\}$. The intersection size depends on j:

- $j = 0$: the two sets are identical, so all three indices overlap. Thus

$$\mathrm{Cov}(Y_n, Y_n) = \mathrm{Var}(Y_n) = \mathrm{Var}(X_n + X_{n+1} + X_{n+2}) = 3\sigma^2.$$

- $j = 1$: the sets are $\{n, n+1, n+2\}$ and $\{n+1, n+2, n+3\}$, whose intersection is $\{n+1, n+2\}$ (two common terms). Hence

$$\mathrm{Cov}(Y_n, Y_{n+1}) = 2\sigma^2.$$

- $j = 2$: the sets are $\{n, n+1, n+2\}$ and $\{n+2, n+3, n+4\}$, with intersection $\{n+2\}$ (one common term). Thus

$$\mathrm{Cov}(Y_n, Y_{n+2}) = \sigma^2.$$

- $j \geq 3$: the sets $\{n, n+1, n+2\}$ and $\{n+j, n+j+1, n+j+2\}$ are disjoint, so there are no common indices and every covariance term is zero. Hence

$$\mathrm{Cov}(Y_n, Y_{n+j}) = 0, \qquad j \geq 3.$$

Therefore,

$$\boxed{\operatorname{Cov}(Y_n, Y_{n+j}) = \begin{cases} 3\sigma^2, & j = 0, \\ 2\sigma^2, & j = 1, \\ \sigma^2, & j = 2, \\ 0, & j \geq 3. \end{cases}}$$

Problem 7.42

Problem. The joint density function of X and Y is given by

$$f_{X,Y}(x, y) = \frac{1}{y} e^{-(y+x/y)}, \qquad x > 0,\ y > 0.$$

Find $E[X]$, $E[Y]$, and show that $\operatorname{Cov}(X, Y) = 1$.

Solution.

1. Find the marginal density of Y

We integrate out x:

$$f_Y(y) = \int_0^\infty f_{X,Y}(x, y)\, dx = \int_0^\infty \frac{1}{y} e^{-(y+x/y)}\, dx, \qquad y > 0.$$

Let $u = x/y$ so that $x = yu$ and $dx = y\, du$. Then

$$f_Y(y) = \frac{1}{y} e^{-y} \int_0^\infty e^{-u} y\, du = e^{-y} \int_0^\infty e^{-u}\, du = e^{-y}, \qquad y > 0.$$

Thus Y has an Exp(1) distribution: $f_Y(y) = e^{-y}$ for $y > 0$, so

$$E[Y] = 1, \qquad \operatorname{Var}(Y) = 1.$$

2. Conditional distribution of X given $Y = y$

For $x > 0, y > 0$,

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{\frac{1}{y} e^{-(y+x/y)}}{e^{-y}} = \frac{1}{y} e^{-x/y}, \qquad x > 0.$$

This is an exponential density with rate $1/y$ (or mean y). Hence

$$X \mid Y = y \sim \operatorname{Exp}\big(\text{rate } = 1/y\big),$$

so

$$E[X \mid Y = y] = y, \qquad \operatorname{Var}(X \mid Y = y) = y^2.$$

Therefore, as random variables,

$$E[X \mid Y] = Y.$$

3. Compute $E[X]$

Using the law of total expectation,

$$E[X] = E\big(E[X \mid Y]\big) = E[Y] = 1.$$

Thus we have

$$E[X] = 1, \qquad E[Y] = 1.$$

4. Show that $\mathrm{Cov}(X, Y) = 1$

We use the identity

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

First compute $E[XY]$ by using the law of total expectation via conditioning on Y:

$$E[XY] = E(E[XY \mid Y]) = E\big(Y\, E[X \mid Y]\big) = E(Y \cdot Y) = E[Y^2].$$

Since $Y \sim \mathrm{Exp}(1)$, we know

$$E[Y^2] = \mathrm{Var}(Y) + (E[Y])^2 = 1 + 1^2 = 2.$$

Therefore,

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y] = 2 - (1)(1) = 1.$$

Hence

$$\boxed{E[X] = 1, \quad E[Y] = 1, \quad \mathrm{Cov}(X, Y) = 1.}$$

Alternatively, if not using conditional expectation, $E(X)$, $E(Y)$, and $E(XY)$ can also be calculated by double integration as shown below:

$$\begin{aligned}
E(X) &= \int_0^\infty \int_0^\infty x f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty \frac{x}{y} e^{-(y+x/y)}\,dx\,dy = \int_0^\infty \frac{e^{-y}}{y} \int_0^\infty x e^{-x/y}\,dx\,dy \\
&= \int_0^\infty \frac{e^{-y}}{y} \left[-xye^{-x/y}\Big|_0^\infty + y \int_0^\infty e^{-x/y}\,dx \right] dy \\
&= \int_0^\infty e^{-y} \int_0^\infty e^{-x/y}\,dx\,dy = \int_0^\infty y e^{-y}\,dy = -ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y} dy = 1.
\end{aligned}$$

$$\begin{aligned}
E(Y) &= \int_0^\infty \int_0^\infty y f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty e^{-(y+x/y)}\,dx\,dy = \int_0^\infty e^{-y} \left(\int_0^\infty e^{-x/y} dx \right) dy \\
&= \int_0^\infty y e^{-y} dy = -ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y} dy = 1.
\end{aligned}$$

$$\begin{aligned}
E(XY) &= \int_0^\infty \int_0^\infty xy\, f(x,y)\, dx\, dy = \int_0^\infty \int_0^\infty xe^{-(y+x/y)} dx\, dy = \int_0^\infty e^{-y} \int_0^\infty xe^{-x/y} dx\, dy \\
&= \int_0^\infty e^{-y} \left[-xye^{-x/y} \Big|_0^\infty + y \int_0^\infty e^{-x/y} dx \right] dy \\
&= \int_0^\infty y^2 e^{-y} dy = -y^2 e^{-y} \Big|_0^\infty + 2 \int_0^\infty ye^{-y} dy = 2.
\end{aligned}$$

$$\text{Hence, } \operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y) = 2 - 1 \cdot 1 = 1.$$

#47. $X_i \sim \Gamma(k,\theta)$, $i=1,2,3$, $Var(X_i)=\frac{k}{\theta^2}\ \forall i$.

since X_i, X_j are uncorrelated $\Rightarrow Cov(X_i,X_j)=0\ \forall i\neq j$

(a) $Cov(X_1,X_1+X_2)=Cov(X_1,X_1)+Cov(X_1,X_2)=Var(X_1)+0=\frac{k}{\theta^2}$

$Var(X_1+X_2)=Var(X_1)+Var(X_2)+2Cov(X_1,X_2)=\frac{k}{\theta^2}+\frac{k}{\theta^2}+0=\frac{2k}{\theta^2}$

$$\Rightarrow Corr(X_1,X_1+X_2)=\frac{Cov(X_1,X_1+X_2)}{\sqrt{Var(X_1)Var(X_1+X_2)}}=\frac{\frac{k}{\theta^2}}{\sqrt{\frac{k}{\theta^2}\cdot\frac{2k}{\theta^2}}}=\frac{1}{\sqrt{2}}$$

(b) $Cov(X_1+2X_2,X_1+X_2+X_3)=Var(X_1)+Cov(X_1,X_2)+Cov(X_1,X_3)+2Cov(X_2,X_1)$

$+2Var(X_2)+2Cov(X_2,X_3)$

$=\frac{k}{\theta^2}+0+0+0+\frac{2k}{\theta^2}+0=\frac{3k}{\theta^2}$

$Var(X_1+2X_2)=Var(X_1)+Var(2X_2)+Cov(X_1,2X_2)=\frac{k}{\theta^2}+\frac{4k}{\theta^2}=\frac{5k}{\theta^2}$

$Var(X_1+X_2+X_3)=Var(X_1)+Var(X_2)+Var(X_3)=\frac{k}{\theta^2}+\frac{k}{\theta^2}+\frac{k}{\theta^2}=\frac{3k}{\theta^2}$

$$Corr(X_1+2X_2,X_1+X_2+X_3)=\frac{Cov(X_1+2X_2,X_1+X_2+X_3)}{\sqrt{Var(X_1+2X_2)Var(X_1+X_2+X_3)}}=\frac{\frac{3k}{\theta^2}}{\sqrt{\frac{5k}{\theta^2}\cdot\frac{3k}{\theta^2}}}$$

$$=\frac{3}{\sqrt{15}}$$

#49. (a) D_i是第i個頂點和其他$(n-1)$個頂點連線成功的總次數、頂點i和不同的頂點連線成功的機率皆為p，且每次連線皆獨立

$\Rightarrow$ independent Bernoulli(p)

$\Rightarrow D_i$是$(n-1)$個independent Bernoulli(p) random variables的總和

$\Rightarrow D_i\sim Bin(n-1,p)$, $Var(D_i)=(n-1)p(1-p)$

(b) 若由n個頂點中任取2個(第k和第l)，其中$k<l$, $k,l=1,2,\cdots,n$

令 $E_{k,l}=\begin{cases}1, & \text{if } k\text{和}l\text{之間有連線}\\ 0 & o.w\end{cases}$

$\Rightarrow E_{k,l}\sim$ Bernoulli(p), $E_{k,l}$'s are independent

#49 (b) $D_i = \sum_{k<i} E_{k,i} + \sum_{i<l} E_{i,l}$, $D_j = \sum_{k<j} E_{k,j} + \sum_{j<l} E_{j,l}$

$i<j$

$$\begin{aligned}Cov(D_i, D_j) &= Cov\Big(\sum_{k<i} E_{k,i} + \sum_{i<l} E_{i,l}, \sum_{k<j} E_{k,j} + \sum_{j<l} E_{j,l}\Big)\\ &= Cov\Big(\sum_{k<i} E_{k,i}, \sum_{k<j} E_{k,j} + \sum_{j<l} E_{j,l}\Big) + Cov\Big(\sum_{i<l} E_{i,l}, \sum_{k<j} E_{k,j} + \sum_{j<l} E_{j,l}\Big)\\ &= Cov(E_{i,j}, E_{i,j}) = Var(E_{i,j}) = p(1-p)\end{aligned}$$

$Var(D_i) = Var(D_j) = (n-1)p(1-p)$

$$\Rightarrow Corr(D_i, D_j) = \frac{Cov(D_i, D_j)}{\sqrt{Var(D_i)Var(D_j)}} = \frac{p(1-p)}{(n-1)p(1-p)} = \frac{1}{n-1}$$

$$\Rightarrow Corr(D_i, D_j) = \begin{cases} \dfrac{1}{n-1} & i \neq j \\ 1 & \forall i = j \end{cases}$$

#53. $$f_X(x) = \begin{cases} \int_0^x 10x^2 y\,dy = 5x^2y^2\Big|_0^x = 5x^4 & 0<x<1 \\ 0 & \text{o.w} \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{10x^2y}{5x^4} = \frac{2y}{x^2}, \quad 0<y<x<1$$

$$E(Y^k|X=x) = \int_0^x y^k \cdot \frac{2y}{x^2}\,dy = \int_0^x \frac{2y^{k+1}}{x^2}\,dy = \frac{2}{k+2}\Big(\frac{y^{k+2}}{x^2}\Big)\Big|_0^x$$
$$= \frac{2}{k+2}x^k, \quad 0<x<1$$

#57. N：一群鴨子裡有N隻鴨子, $N\sim Poisson(6)$. 10位獵人的目標互相獨立, 10位獵人的命中率是0.6.

$\forall 1\le i\le N$. $X_i=\begin{cases}1 & \text{第}i\text{隻鴨子被命中}\\ 0 & \text{第}i\text{隻鴨子沒有被命中}\end{cases}$

⇒令X是被命中的鴨子總數, 則 $X=\sum_{i=1}^{N}X_i$

Given $N=n$. $\forall 1\le i\le n$, $X_i|n\sim Bernoulli(p)$

$p=P(X_i=1|N=n)=1-P(\text{沒有人命中第}i\text{隻鴨子})=1-\prod_{j=1}^{10}(\underbrace{\tfrac{1}{n}\cdot 0.4}_{\text{第}j\text{個人選到第}i\text{隻鴨子但沒命中}}+\underbrace{\tfrac{n-1}{n}\cdot 1}_{\text{第}j\text{個人沒選到第}i\text{隻鴨}})$

$=1-(1-\frac{0.6}{n})^{10}$

$E(X|N=n)=E(\sum_{i=1}^{n}X_i|N=n)=\sum_{i=1}^{n}E(X_i|N=n)=\sum_{i=1}^{n}p=np=n\cdot(1-(1-\frac{0.6}{n})^{10})$

$E(X)=E_N(E(X|N))=E(N\cdot(1-(1-\frac{0.6}{N})^{10}))=\sum_{n=0}^{\infty}n\cdot(1-(1-\frac{0.6}{n})^{10})\cdot\frac{e^{-6}6^n}{n!}$

Rmk: (1) 計算不需使用 $X_1,\dots,X_n$ 在給定 $N=n$ 時的joint pmf, 只需使用各自的marginal pmf, 這使得計算可以簡化

(2) 因為在給定 $N|n$ 時 $X_1,\dots,X_n$ 並非independent ⇒ $X|N=n$ 的分配不是 $Binomial(n,p)$

#70 X：明年的風暴次數

$Y=\begin{cases}1 & \text{明年是good year}\\ 0 & \text{明年是bad year}\end{cases}$ ⇒ $Y\sim Ber(0.4)$

$X|Y=1\sim Poisson(3)$, $X|Y=0\sim Poisson(5)$

$E(X|Y=1)=3=Var(X|Y=1)$, $E(X|Y=0)=5=Var(X|Y=0)$

$E(X)=E_Y(E(X|Y))=0.4\cdot E(X|Y=1)+0.6\,E(X|Y=0)=4.2.$

$Var(X)=E_Y(Var(X|Y))+Var_Y(E(X|Y))$

$=0.4\cdot Var(X|Y=1)+0.6\cdot Var(X|Y=1)+[0.4\cdot(3-4.2)^2+0.6\cdot(5-4.2)^2]$

$=0.4\cdot 3+0.6\cdot 5+0.96=5.16$

NTHU MATH 2810, 2025 **Solution to Homework 10**

Problems:

7.79

$$X \sim U(0,1), \quad a_0 = 0,\ a_1 = \frac{1}{2},\ a_2 = 1.$$

$$I = \begin{cases} 0, & a_0 = 0 < X \le \frac{1}{2} = a_1, \\ 1, & a_1 = \frac{1}{2} < X \le 1 = a_2. \end{cases}$$
$$\Longrightarrow I \sim \text{Bernoulli}(\frac{1}{2})$$

$$f_{X|I}(x|I=0) = \begin{cases} \frac{1}{P(0<X\le\frac{1}{2})} = 2, & 0 < x \le \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{X|I}(x|I=1) = \begin{cases} \frac{1}{P(\frac{1}{2}<X\le 1)} = 2, & \frac{1}{2} < x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. By Proposition 6.1, the quantity is minimized when

   $$y_i = E(X \mid I = i) = E(X \mid a_i < X \le a_{i+1}), \quad i = 0, 1.$$

   So, we have

   $$y_0 = E(X \mid I = 0) = \int_0^{1/2} x f_{X|I=0}(x)\ dx = \int_0^{1/2} 2x\,dx = \frac{1}{4},$$
   $$y_1 = E(X \mid I = 1) = \int_{1/2}^{1} x f_{X|I=1}(x)\ dx = \int_{1/2}^{1} 2x\,dx = \frac{3}{4}.$$

   Now, the optimal quantizer is given by $Y = E[X|I]$.

2. Calculate $E[(X-Y)^2]$

   $$\begin{aligned} E\Big[(X-Y)^2\Big] &= \sum_{i=0}^{1} E\Big[(X-y_i)^2 \mid I = i\Big]\, P(I=i) \\ &= \frac{1}{2}\left(\int_0^{1/2} 2\Big(x - \frac{1}{4}\Big)^2 dx + \int_{1/2}^{1} 2\Big(x - \frac{3}{4}\Big)^2 dx\right) \\ &= \frac{1}{48}. \end{aligned}$$

7.82

By question, we have the joint pdf of X and Y s.t.

$$f_{X,Y}(x,y)=\frac{1}{\sqrt{2\pi}}e^{-y}\exp\left[-\frac{(x-y)^2}{2}\right],\quad 0<y<\infty,\ -\infty<x<\infty.$$

(a)

$$\begin{aligned}
M_{X,Y}(t_1,t_2) &= E\big(e^{t_1X+t_2Y}\big)\\
&= \int_0^\infty\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{t_1x+t_2y}e^{-y}\exp\left[-\frac{(x-y)^2}{2}\right]\,dx\,dy\\
&= \int_0^\infty\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[t_1x+t_2y-y-\frac{x^2-2xy+y^2}{2}\right]\,dx\,dy\\
&= \int_0^\infty\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\big(x^2-2(t_1+y)x\big)+t_2y-y-\frac{y^2}{2}\right]\,dx\,dy\\
&= \int_0^\infty\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\big(x-(t_1+y)\big)^2+\frac{(t_1+y)^2}{2}+t_2y-y-\frac{y^2}{2}\right]\,dx\,dy\\
&= \int_0^\infty \exp\left[\frac{t_1^2}{2}+(t_1+t_2-1)y\right]\int_{-\infty}^\infty \underbrace{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\big(x-(t_1+y)\big)^2\right]}_{\text{pdf of } N(y+t_1,1)\ \Longrightarrow\ \text{積分值}=1}dx\,dy\\
&= e^{t_1^2/2}\int_0^\infty e^{(t_1+t_2-1)y}\,dy\\
&= \begin{cases}\frac{e^{t_1^2/2}}{1-t_1-t_2}, & t_1+t_2<1,\\ \text{do not exist}, & t_1+t_2\ge 1.\end{cases}
\end{aligned}$$

(b)

$$M_X(t_1)=M_{X,Y}(t_1,0)=\begin{cases}\frac{e^{t_1^2/2}}{1-t_1}, & t_1<1,\\ \text{do not exist}, & t_1\ge 1.\end{cases}$$

$$M_Y(t_2)=M_{X,Y}(0,t_2)=\begin{cases}\frac{1}{1-t_2}, & t_2<1,\\ \text{do not exist}, & t_2\ge 1.\end{cases}$$

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Solution Manual **Homework 10**
NTHU MATH 2810, Probability (Sep ∼ Dec 2025) Term: Fall 2025
Instructor: Shao-Wei Cheng Date: December 10, 2025

Theoretical 7.6.
Let $A_1, A_2, \ldots, A_n$ be arbitrary events, and define $C_k = \{\text{at least } k \text{ of the } A_i \text{ occur}\}$. Show that

$$\sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P(A_k)$$

Hint: Let X denote the number of the A_i that occur. Show that both sides of the preceding equation are equal to $E[X]$.

Solution:

令 X 表示 $A_1, \ldots, A_n$ 中發生的事件總數。我們可以利用 Indicator 來表示 A_i：

$$I_{A_i} = \begin{cases} 1 & 若 A_i \text{ 發生} \\ 0 & 若 A_i \text{ 未發生} \end{cases} \quad \Rightarrow \quad I_{A_i} \sim \text{Bernoulli}(P(A_i))$$

則 X 可寫為所有 I_{A_i} 的總和：

$$X = \sum_{i=1}^{n} I_{A_i}$$

對兩邊取期望值：

$$E[X] = E\left[\sum_{i=1}^{n} I_{A_i}\right] = \sum_{i=1}^{n} E[I_{A_i}] = \sum_{i=1}^{n} P(A_i) \quad \ldots (*1)$$

另一方面，由定義可知 $C_k = \{X \geq k\}$，因此：

$$\begin{aligned} \sum_{k=1}^{n} P(C_k) &= \sum_{k=1}^{n} P(X \geq k) \\ &= \sum_{k=1}^{n} \sum_{s=k}^{n} P(X = s) \\ &= \sum_{s=1}^{n} \sum_{k=1}^{s} P(X = s) \\ &= \sum_{s=1}^{n} s \cdot P(X = s) \\ &= \sum_{s=0}^{n} s \cdot P(X = s) \quad (\text{補上} s = 0 \text{ 的項，其值為 } 0) \\ &= E[X] \quad \ldots (*2) \end{aligned}$$

由 $(*1)$ 和 $(*2)$ 可知：

$$\sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P(A_k)$$

註: $(*2)$ 其實為套用 Homework 4 中 **Theoretical Exercises — Problem 4.5** 之結果。

Theoretical 7.18.
Suppose that X_1 and X_2 are independent random variables having a common mean μ. Suppose also that $\mathrm{Var}(X_1) = \sigma_1^2$ and $\mathrm{Var}(X_2) = \sigma_2^2$. The value of μ is unknown, and it is proposed that μ be estimated by a weighted average of X_1 and X_2. That is, $\lambda X_1 + (1-\lambda)X_2$ will be used as an estimate of μ for some appropriate value of λ. Which value of λ yields the estimate having the lowest possible variance? Explain why it is desirable to use this value of λ.

Solution:
令估計量為 $\hat{\mu} = \lambda X_1 + (1-\lambda)X_2$，期望值為 $E[\hat{\mu}] = \lambda\mu + (1-\lambda)\mu = \mu$ (不偏估計量)。

由於 X_1 與 X_2 獨立，其 $Cov(X_1, X_2) = 0$，故變異數為：

$$\begin{aligned}\mathrm{Var}(\hat{\mu}) &= \lambda^2\mathrm{Var}(X_1) + (1-\lambda)^2\mathrm{Var}(X_2)\\ &= \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2\end{aligned}$$

為了挑選 λ 以使得變異數能最小，我們對 λ 微分並設為 0：

$$\frac{d}{d\lambda}\mathrm{Var}(\hat{\mu}) = 2\lambda\sigma_1^2 + 2(1-\lambda)(-1)\sigma_2^2 = 0$$

$$\begin{aligned}2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2 &= 0\\ \lambda\sigma_1^2 &= (1-\lambda)\sigma_2^2\\ \lambda\sigma_1^2 &= \sigma_2^2 - \lambda\sigma_2^2\\ \lambda(\sigma_1^2+\sigma_2^2) &= \sigma_2^2\\ \lambda &= \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\end{aligned}$$

因二階導數為 $\frac{d^2}{d\lambda^2}\mathrm{Var}(\hat{\mu}) = 2(\sigma_1^2+\sigma_2^2) > 0$，故確認其為最小值。

使用此 λ 值，我們給 X_1 的權重是 $\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$，給 X_2 的權重是 $\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$，這表示變異數越小的變數，其權重越大（權重與變異數成反比）。

另外，由於 $E[\hat{\mu}] = \mu$，故 $\mathrm{Var}(\hat{\mu}) = E\{(\hat{\mu}-\mu)^2\}$ 即為估計量 $\hat{\mu}$ 的均方誤差 (Mean Squared Error, MSE)，其中 $\hat{\mu}-\mu$ 為估計誤差。因此，找最小變異數等價於找最小的 MSE。

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Theoretical 7.52.
Use Table 7.2 to determine the distribution of $\sum_{i=1}^{n} X_i$ when $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables, each having mean $1/\lambda$.

Solution:
已知 $X_i \sim \mathrm{Exp}(\lambda)$ (平均值 $1/\lambda$)，其動差生成函數 (MGF) 為：

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

令 $Y = \sum_{i=1}^{n} X_i$。由於 X_i 互相獨立，Y 的 MGF 等於各個 X_i 的 MGF 皆代入相同的 t 時的乘積：

$$\begin{aligned} M_Y(t) &= \prod_{i=1}^{n} M_{X_i}(t) \\ &= \left(\frac{\lambda}{\lambda - t}\right)^n \end{aligned}$$

查 Table 7.2，這就是 $\mathrm{Gamma}(s = n, \lambda)$ 分布的 MGF 形式。

根據 MGF 的唯一性，可知：$\sum_{i=1}^{n} X_i \sim \mathrm{Gamma}(n, \lambda)$

Table 7.2 Continuous Probability Distribution.

	Probability density function, $f(x)$	Moment generating function, $M(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters $(s, \lambda), \lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^s$	$\frac{s}{\lambda}$	$\frac{s}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Figure 1: Table 7.2